ISSN: 2458-8989

Natural and Engineering Sciences

Supplement, 2019, 4(3): 52-58

Common fixed point theorem for two pairs of weakly compatible mappings satisfying a larger generalized (S,T)-condition

Aouine Ahmed Chaouki[1]

[1] Department of mathematics, university of Souk-ahras 41000, Algeria

[1] Department of mathematics, university of Oum El Bouaghi 04000; Algeria

Emails: chawki81@gmail.com

Abstract

In this paper, we prove a common fixed point theorem for two pairs of weakly compatible mappings satisfying a generalized condition in metric spaces and we present an example which illustrates our results.

Keyword(s): Common fixed point, weakly compatible, generalized condition, metric spaces.

Introduction

Fixed point theory has fascinated hundreds of researchers since 1922 with the celebrated Banach's fixed point theorem. This theorem provides a technique for solving a variety of applied problems in mathematical sciences and engineering. In the study of functional analysis and topology, metric spaces play very important role and gained considerable importance after the famous Banach Contraction Principle.

In recent years, many fixed point theorems have appeared in the literature using the notion of compatibility by various authors. Sessa [9] generalized the concept of commutative mappings by introducing the concept of weakly commutative mappings. Jungck [1] generalized the concept of weak commutativity by introducing the concept of compatible mappings. Jungck and al [2] generalized the concept of compatibility by introducing the concept of compatible mappings of type (A). Pathak et al [5,6,8] generalized the concept of compatibility of type (A) by introducing the concept of compatibility of type (B), the concept of compatibility of type(P) and the concept of compatibility of type (C). It was shown in [2,5,6,8] that these notions are equivalent if the mappings are continuous. In [3], Jungck introduced the concept of weakly compatibility mappings. It was shown in [1,2,5,6,8] that each of these concepts of compatibility implies weakly compatibility, but the converse is not true in general. In other words the weakly compatibility is the lowest among all cited notion compatibility. In the following of this section S and T denote two mappings of a metric space (X,d) into itself.

S and T are said to be commutative if $STx=TSx$ for any $x\in X$.

Materials and Methods

1.1. Definition 1.1 ([5]). S and T are said to be weakly compatible if they commute at coincidence points, i.e., if St = Tt for t ∈ X, then STt = TSt.

Definition 1.2 ([1]). S and T are said to be compatible if

(1.2) $$\lim_{\nu!1} \delta(\Sigma T\xi_\nu; T\Sigma\xi_\nu) = 0;$$

whenever $f\xi_\nu g$ is a sequence in Ξ satisfying

(1.3) $$\lim_{\nu!1} \Sigma\xi_\nu = \lim_{\nu!1} T\xi_\nu = \tau \text{ for a certain } \tau\ 2\ \Xi:$$

It is easy to show that weakly commutativity implies compatibility, but the converse not be true in general as it proved in [4].

Definition 1.3 ([3]). Σ and T are said to be weakly compatible if they commute at coincidence points, i.e., if $\Sigma\tau = T\tau$ for $\tau\ 2\ \Xi$, then $\Sigma T\tau = T\Sigma\tau$.

The purpose of this paper is to present a common .xed point result for four mappings which satisfy larger generalized $(\Sigma; T)$bcontractive condition in metric spaces. For this aim we need the following definitions:

Definition 1.4 Let $(\Xi; \delta)$ be a metric space and K a nonempty subset of Ξ and $\Phi; \Gamma; \Sigma; T : K\ !\ \Xi$ a mappings satisfying:

$$\delta(\Phi\xi; \Gamma\psi) \leq \alpha \max f1/2\delta(T\xi; \Sigma\psi); \delta(T\xi; \Phi\xi); \delta(\Sigma\psi;\Gamma\psi)g$$
$$+\beta f\delta(T\xi; \Gamma\psi) + \delta(\Phi\xi; \Sigma\psi)g$$

for any $\xi; \psi\ 2$ K and $\xi\ 6= \psi; \alpha; \beta \geq 0$ such that $\alpha + 2\beta < 1$: Then $(\Phi;\Gamma)$ is said a generalized $(T; \Sigma)$ contraction in K.

Theorem 1.5.

Let (X,d) be an complete metric space. Let A,B,S and T be self maps on X satisfying the following conditions:

$$A(X)\subset T(X) \text{ et } B(X)\subset S(X) \qquad \#2.1$$

$$[1 + pd(Sx; Ty)]d(Ax,By) \leq p\max\{d(Sx; Ax)d(Ty; By); d(Sx; By)d(Ty; Ax)\} \qquad \#2.2$$
$$+\ni [\delta\max d(Sx,Ty),d(Sx,Ax),d(Ty,By)$$
$$+((d(Sx,By)+d(Ty,Ax))/2)\}+(1-\delta)\max\{d(Ax,Sx),$$
$$d(By,Ty)\}]+L\min\{d(Sx,Ax),d(Ty,By),$$
$$d(Sx,By),d(Ty,Ax)\},$$

for any $\xi; \psi\ 2\ \Xi$**, where** $0 <^{™}\delta 1; 0 < \eta < 1, \Lambda\varepsilon 0, \pi \geq 0$ **and** $\ni : [0; 1)\ !\ [0; 1)$ **is a upper semicontinuous function** with $\ni(t) = 0$ if and only if $t = 0$

(b): for all $\tau > 0$; $\ni(\tau) < 0$ if and only if $\lim_{\nu!1} \ni^{\nu}(\tau) = 0$. Suppose that $\Sigma(\Xi)$ or $T(\Xi)$ is complete and the pairs $(A; \Sigma)$ and $(B; T)$ are weakly compatible, then $A;B; \Sigma$ and T have a unique common fixed point in Ξ.

Proof. *Let x_0 an arbitrary point in X. From (2.1), there exist a point $x_1 \in X$ such that $Ax_0 = Tx_1$: for this point x_1, we can choose a point x_2 such that $Bx_1 = Sx_2$: Inductively, we can define a sequence $\{y_n\}$ in X such that*

$$y_{2n}=Ax_{2n}=Tx_{2n+1} \text{ and } y_{2n+1}=Sx_{2n+2}=Bx_{2n+1},\ n\in N: \qquad \#(2.3)$$

Now, we will show that the sequence $\{y_n\}$ defined above is a Cauchy sequence in X. First suppose that $y_n \neq y_{n+1}$ for any n. we use (2.2) and (2.3) . Let us denote (y_n, y_{n+1}) by n, for each n = 0;1;2... First we will show that $\alpha_{n+1} \lesssim \ni(\alpha_n)$ and then we claim that

$\lim\alpha_n=0$ such that $\nu!1$ #2.4
and then show that $\{y_n\}$ is a Cauchy sequence in X. For this, putting $x=x_{2n+2}$ and $y=x_{2n+1}$ in (2.2) we obtain

$[1+p\alpha_{2n}]\alpha_{2n+1}\}\leq p\max\{\alpha_{2n+1}\alpha_{2n},0\}+\phi(\delta\max\{\alpha_{2n},\alpha_{2n+1},\alpha_{2n},(1/2)\alpha(y_{2n},y_{2n+2})\}+(1-\delta)\max\{\alpha_{2n+1},\alpha_{2n}\})$

But, from the triangle inequality for metric α, we have

$(1/2)\alpha(y_{2n},y_{2n+2})\leq (1/2)[\alpha(y_{2n},y_{2n+2})+\alpha(y_{2n+1},y_{2n+2})]=(1/2)(\alpha_{2n},\alpha_{2n+1})$

$\leq \max\{\alpha_{2n},\alpha_{2n+1}\}.$

Using this in above, we get

$[1+p\alpha_n]\alpha_{2n+1}\leq p\max\{\alpha_{2n+1}\alpha_{2n}\}+\phi(\delta\max\{\alpha_{2n}\alpha_{2n+1}\}+(1-\delta)\max\{\alpha_{2n+1},\alpha_{2n}\}))$.

If we choose α_{2n+1} as "max" in above then we obtain

$\alpha_{2n+1}\leq\phi(\alpha_{2n+1})<\alpha_{2n+1}$,

a contradiction. Hence,

$\alpha_{2n+1}\leq\phi(\alpha_{2n})$. #2.5

Similarly, by setting x_{2n+2} for x and x_{2n+3} for y in (2.2) we have

$[1+p\alpha_{2n+1}]\alpha_{2n+2}\leq p\max\{\alpha_{2n+1}\alpha_{2n+2},0\}+\phi(\max\{\alpha_{2n+1},\alpha_{2n+1},\alpha_{2n+2},(1/2)\alpha(y_{2n+1},y_{2n+3})\})$,

i.e.,

$\alpha_{2n+2}\leq\phi(\max\{\alpha_{2n+1},\alpha_{2n+1},\alpha_{2n+2},(1/2)\alpha(y_{2n+1},y_{2n+3})\})=\phi(\alpha_{2n+1})$,

hence

$\alpha_{2n+2}\leq\phi(\alpha_{2n+1})$. #2.6

Unifying (2.5) and (2.6) we obtain

$\alpha_{n+1}\leq\phi(\alpha_n)$,

which implies that

$\alpha_n\leq\phi(\alpha_{n-1})\leq\phi^2(\alpha_{n-2})\leq...\leq\phi^n(\alpha_o)$,

and by condition (a) and (b) in theorem (2.1) since $\lim\phi^n(\alpha_o)=0$ if $\alpha_o=0$, we have

$\lim\alpha_n=0$, such that $\nu!1$
thus $\{y_n\}$ is a Cauchy sequence and since X is complete, there exists a point z in X such that $\lim y_n = z$. The sequence $\{y_{2n+1}\}=\{Sx_{2n+2}\}\subset S(X)$ is a Cauchy sequence in S(X). Suppose that S(X) is complete. Then it converges to a point $z=Su$ for $u\in X$. Therefore, the subsequences $\{Ax_{2n}\}$, $\{Bx_{2n+1}\}$, $\{Tx_{2n+1}\}$ also converge to z. If $Au\neq z$, using (2.2) we get

$[1+d(Su,Tx_{2n+1})]d(Au,Bx_{2n+1}) \leq p\max\{d(Su,Au)d(Tx_{2n+1},Bx_{2n+1}),d(Bx_{2n+1},Su)d(Au,Tx_{2n+1})\}$

$+\phi[\delta\max\{d(Su,Tx_{2n+1}),d(Su,Au),d(Tx_{2n+1},Bx_{2n+1}),$

$((d(Su,Bx_{2n+1})+d(Tx_{2n+1},Au))/2)\}+(1-\delta)\max\{d(Au,Su),$

$d(Tx_{2n+1},Bx_{2n+1})\}]+L\min\{d(Su,Tx_{2n+1})d(Su,Au),d(Tx_{2n+1},Bx_{2n+1}),$

$d(Su,Bx_{2n+1}),d(Tx_{2n+1},Au)\}.$

Letting $n\to\infty$ we obtain

$$d(Au,z) \le \phi[\delta d(Au,z)+(1-\delta)d(Au,z)]$$

$$< d(Au,z),$$

which is impossible. Hence, $z=Au=Su$. Since $A(X)\subset T(X)$, there exists $v\in X$ such that $z=Tv$. If $z\neq Bv$, Applying (2.2) we have

$$[1+d(Su,Tv)]d(Au,Bv) \le p\max\{d(Su,Au)d(Tv,Bv),d(Bv,Su)d(Au,Tv)\}$$

$$+\phi[\delta\max\{d(Su,Tv),d(Su,Au),d(Tv,Bv),$$

$$((d(Su,Bv)+d(Tv,Au))/2)\}+(1-\delta)\max\{d(Au,Su),d(Tv,Bv)\}]$$

$$+L\min\{d(Su,Tv),d(Su,Au),d(Tv,Bv),d(Su,Bv),d(Tv,Au)\}.$$

Then

$$d(z,Bv) \le \phi[\delta d(z,Bv)+(1-\delta)d(z,Bv)]$$

$$< d(z,Bv),$$

which is impossible. Therefore, $z=Bv=Tv$. As (A,S) is weakly compatible, we find $SAu=ASu$, i.e., $Az=Sz$. If $Az\neq z$, using (2.2) we get

$$[1+d(Sz,Tv)]d(Az,Bv) \le p\max\{d(Sz,Az)d(Tv,Bv),d(Bv,Sz)d(Az,Tv)\}$$

$$+\phi[\delta\max\{d(Sz,Tv),d(Sz,Az),d(Tv,Bv),$$

$$((d(Sz,Bv)+d(Tv,Az))/2)\}+(1-\delta)\max\{d(Az,Sz),d(Tv,Bv)\}]$$

$$+L\min\{d(Sz,Tv),d(Sz,Az),d(Tv,Bv),d(Sz,Bv),d(Tv,Az)\},$$

then

$$d(z,Az) \le \phi[\delta d(Az,z)+(1-\delta)d(Az,z)]$$

$$< d(z,Az),$$

which is impossible. So, $z=Az=Sz$. Similarly, we can prove that $z=Bz=Tz$. Assume there exists n such that $y_{n}=y_{n+1}$. By induction, $y_{n}=y_{n+k}$ for $k\geq 1$. Thus, there exists $u,v\in X$ such that $Au=Su$ et $Bv=Tv$. We can prove that $z=Az=Sz=Bz=Tz$. For the uniqueness of z, suppose that w is another common fixed point of A,B,S and T. Applying (2.2) we ob tain

$$d(Az,Bw) = d(z,w) \le \phi\delta d(z,w)<d(z,w),$$

which is impossible. Hence, $z=w$. Then, A,B,S and T have a unique common fixed point in X.

Results and Discussion

Example

Let A,B,S and T be four self-mappings of a metric space X, endowed with the usual metric d. Let $X=[0,3/2]$. Define the mappings A,B,S and $T:X\rightarrow X$ by:

$$Ax=1,Sx=x,Bx=1 \text{ and } Tx=1/2(1+x);\forall x\in X.$$

Let $\phi:[0;1)\rightarrow\mathbb{R}$ be defined by $\phi(t)=t/2$. Then we observe that:

(a) $AX=\{1\}\subseteq TX=[1/2,5/4]\subseteq X$ and $BX=\{1\}\subseteq SX=[0,3/2]\subseteq X$,

(b) Since, $d(Ax,By)=0$, $d(Sx,Ty)=1/2|2x-y-1|$,$d(Ax,Sx)=|1-x|$,$d(By,Ty)=|1-y|=d(Ax,Ty)$ and $d(By,Sx)=|1-x|,\forall x,y\in X$, we have for condition (2.2)

that

$$\begin{aligned}[1+pd(Sx,Ty)]d(Ax,By) \leq\ & p\max\{d(Sx,Ax)d(Ty,By),d(Sx,By)d(Ty,Ax)\} \\ & +\phi[\delta\max\{d(Sx,Ty),d(Sx,Ax),d(Ty,By), \\ & ((d(Sx,By)+d(Ty,Ax))/2)\}+(1-\delta)\max\{d(Ax,Sx),d(By,Ty)\}] \\ & +L\min\{d(Sx,Ty),d(Sx,Ax),d(Ty,By),d(Sx,By),d(Ty,Ax)\},\end{aligned}$$

or

$$\begin{aligned}0 \leq\ & p/2|1-x||1-y|+\phi[\delta\max\{1/2|2x-y-1|,|1-x|,|1-y|, \\ & ((|1-x|+|1-y|)/2)\}+(1-\delta)\max\{|1-x|,|1-y|\}] \\ & +L\min\{1/2|2x-y-1|,|1-x|,|1-y|,|1-x|,|1-y|\},\end{aligned}$$

where the socend membre is positif. Thus condition (2.2) is true for all $\forall x,y\in X$ and $p\geq 0$.Further, we see that

$$M(x,y)=\delta\max\{1/2|2x-y-1|,|1-x|,|1-y|,((|1-x|+|1-y|)/2)\}+(1-\delta)\max\{|1-x|,|1-y|\}=0,$$

if and only if,

$$1/2|2x-y-1|=|1-x|=|1-y|=((|1-x|+|1-y|)/2)=0,$$

i.e., $x=1,y=1$. Thus $M(1,1)=0$ and therefore $\phi(0)=0$.We notice that the pairs (A,S) and (B,T) have the coincidence point $x=1$ where they commutes. So that (A,S) and (B,T) are weakly compatible. Thus all the conditions of Theorem 2.1 are satisfied. Hence $x=1$ is the unique common fixed point of A,B,S and T

Conclusion

If B=A and T=S in the Theorem 2.1, we get the following corollary.

Corollary 2.2. Let A and S be two mappings of a metric space (X,d) into itself satisfying

$A(X)\subset S(X)$

$$[1+pd(Sx,Sy)]d(Ax,Ay) \leq p\max\{d(Sx,Ax)d(Sy,Ay),d(Sx,Ay)d(Sy,Ax)\} \quad \#2.7$$
$$+\phi[\delta\max d(Sx,Sy),d(Sx,Ax),d(Sy,Ay),$$
$$((d(Sx,Ay)+d(Sy,Ax))/2)\}+(1-\delta)\max\{d(Ax,Sx),d(Ay,Sy)\}]$$
$$+L\min\{d(Sx,Sy),d(Sx,Ax),d(Sy,Ay),d(Sx,Ay),d(Sy,Ax)\}.$$

for any $x,y\in X$, where $0<\delta\leq 1$, $L\geq 0$ and

(a) $\phi : [0, \infty) \to [0, \infty)$ is a upper semicontinuous function with $\phi(t)=0$ if and only if $t=0$,

(b) for all $t>0,\phi(t)<0$ if and only if $\lim\phi^n(t)=0$ Suppose that S(X) is complete and (A,S) is weakly compatible. Then, A and S have a unique common fixed point in X.

If $S=I_X$ in the corollary 2.2, where I_X is identity mapping in X, then we obtain the following corollary.

Corollary 2.3. Let A a mapping of a Banach space (X,d) into itself satisfying

$$[1+pd(x,y)]d(Ax,Ay) \leq p\max\{d(x,Ax)d(y,Ay),d(x,Ay)d(y,Ax)\} \quad \#2.8$$
$$+\phi[\delta\max d(x,y),d(x,Ax),d(y,Ay),$$
$$((d(x,Ay)+d(y,Ax))/2)\}+(1-\delta)\max\{d(Ax,x),d(Ay,y)\}]$$
$$+L\min\{d(x,y),d(x,Ax),d(y,Ay),d(x,Ay),d(y,Ax)\}.$$

for any $x,y\in X$, where $0<\delta\leq 1$, and $L\geq 0$ and

(a) $\phi : [0, \infty) \to [0, \infty)$ is a upper semicontinuous function with $\phi(t)=0$ if and only if $t=0$,

(b) for all $t>0,\phi(t)<0$ if and only if $\lim\phi^n(t)=0$ Then, A has a unique fixed point in X.

Acknowledgements

Thanks for all.

Conflict of Interest: The authors declare that they have no conflict of interest.

Note: This paper is presented in the International Conference on Artificial Intelligence towards Industry 4.0 held on November 14 - 16, 2019 at Iskenderun Technical University, Iskenderun, Turkey.

References

Jungck G. (1986). Compatible mappings and common fixed points. Internat J. Math. Math. Sci., 9, 127-179.

Jungck G., Murthy P. P. and Cho Y. J. (1993). Compatible mappings of type (A) and common fixed points, Math. Japonica., 38 (2), 381-390.

Jungck G. (1996). Common fixed points for non-continuous non-self maps on non metric spaces, Far East J. Math. Sci., 4 (2), 199-215.

Matkowski J. (1975). Integrable solutions of functional equations. Diss. Math., 127.

Pathak H. K and Khan M. S. (1995). Compatible mappings of type (B) and common fixed point theorems of Gregus type, Czechoslovak Math. J., 45 (120), 685-698.

Pathak H. K., Cho Y. J., Kang S. M., Lee B. S. (1995). Fixed point theorems for compatible mappings of type (P) and applications to dynamic programming, Le Matematiche., Fasc. I, 50, 15.33.

Pathak H. K., Cho Y. J., Chang S., Kang, S. M. (1996). Compatible mappings of type (P) and fixed point theorem in metric spaces and probabilistic metric spaces, Novi Sad J. Math., 26 (2), 87-109.

Pathak H. K., Cho Y. J., Khan S. M. and Madharia B. (1998). Compatible mappings of type (C) and common fixed point theorems of Gregus type, Demonstratio Math., 31 (3), 499-518.

Sessa S. (1982). On a weak commutativity condition of mappings in fixed point considerations. Publ. Inst. Math. Beograd., 32 (46), 149-153.